



Linear Quadratic Gaussian Homing for Markov Processes with Regime Switching and Applications to Controlled Population Growth/Decay

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Received: 30 April 2019 / Revised: 21 February 2020 / Accepted: 3 June 2020 /

Published online: 17 July 2020

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Abstract

The problem of optimally controlling one-dimensional diffusion processes until they enter a given stopping set is extended to include Markov regime switching. The optimal control problem is presented by making use of dynamic programming. In the case where the Markov chain has two states, the optimal homotopy analysis method (OHAM) is used to obtain an analytical approximation of the value function, which is compared to the finite difference approximation with successive updates of the nonlinear and coupling terms. As an example, the method is applied to controlled population growth with regime switching.

Keywords Markov chain · Regime switching · Optimal homotopy analysis method · Linear quadratic Gaussian · Optimal control · Viscosity solution

Mathematics Subject Classification (2010) MSC 49 · MSC 60 · MSC 65

1 Introduction

The linear quadratic Gaussian (LQG) homing, optimal control problem was described in Whittle and Gait (1970) to minimize the cost incurred up to a time τ for the first entry into a termination set D . Recently, Lefebvre (2014) successfully extended the optimal control of one-dimensional diffusion processes entering a given stopping set to the case of jump-diffusion processes.

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In this paper we further extend the problem by allowing the drift and standard deviation of the controlled process to change based on the state of a Markov chain, which defines the regime switching terminology. After Hamilton's seminal work, Hamilton (1989) regime changes have found applications in biology (Li et al. 2009; Liu and Wang 2010; Luo and Mao 2007, 2009) and finance (Asmussen 1989; Lu and Li 2005).

Bacterial growth is related to this work and previously discussed by Hieber (2014). As an example, consider a bacteria whose growth rate follows the regime switching model. If the temperature is between T_1 and T_2 , the growth rate is described by a Brownian motion with parameters (μ_1, σ_1) ; however, when the temperature rises above T_2 , the growth rate is described by (μ_2, σ_2) .

As another example, consider the population growth of cancer cells given by the stochastic Gompertz model, Ferrante et al. (2000) which can be written as a controlled stochastic process. The size of the tumor depends on both the growth rate and control rate. This process can be controlled until the tumor size is decreased below a critical value. The model parameters μ_k and σ_k can be assumed to change, e.g., multiple tissues or drug regimens.

We determine the optimal control for Brownian-like motions with regime switching. We derive an analytical approximation for the value function of the uncontrolled process that satisfies a system of nonlinear coupled equations using optimal homotopy analysis method (OHAM). This analytical method is compared to a successive iteration, finite-difference approximation.

2 Notation and Problem Formulation

2.1 Notation

Through out this paper we are going to use the following notation:

1. $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space; $b_0 \in \mathbb{R}^*$;
2. $a \wedge b = \min(a, b)$; if $i \in \{1, 2, \dots, n\}$ we denote by $\bar{i} \in \{1, 2, \dots, n\}$ such $i \neq \bar{i}$;
3. $\{W(t) : t \in [s, T]\}$ is a one-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathcal{P})$ over $[s, T]$;
4. $\{\alpha(t) : t \in [s, T]\}$ is a continuous time Markov process on $(\Omega, \mathcal{F}, \mathcal{P})$ with finite state space $\mathbb{M} = \{1, 2, 3, \dots, m\}$
 $-\mathcal{P}(\alpha(t+dt) = l \mid \alpha(t) = k) = q_{kl}dt + o(dt) \quad \text{if } k \neq l$
 $-\mathcal{P}(\alpha(t+dt) = l \mid \alpha(t) = k) = 1 + q_{kk}dt + o(dt) \quad \text{if } k = l$
 $\sum_{j=1}^m q_{ij} = 0$;
5. $\mathcal{F}_t = \sigma\{W(s), \alpha(s) : 0 \leq s \leq t\}$ and $W(t), \alpha(t)$ independent;
6. the control $\underline{u} : \bar{Q}_s \times \mathbb{M} \rightarrow U$ is an $\{\mathcal{F}_t\}_{t \geq s}$ -adapted process on $(\Omega, \mathcal{F}, \mathcal{P})$ where $\bar{Q}_s = [s, T] \times [a, b]$, for simplicity, we set $u(t) = \underline{u}(t, x, k)$;
7. $\mathcal{U}^0(s) = L^\infty([s, T]; U)$ is the space of all bounded, Lebesgue measurable, U -valued functions on $[s, T]$

$$\mathcal{U}(s, x) = \{u(\cdot) \in \mathcal{U}^0(s) : x(t) \in \bar{Q}_s\};$$
8. a function $\varphi(\cdot, \cdot, \cdot)$ on $\bar{Q}_s \times \mathbb{M}$ satisfies the polynomial growth condition; if for some positive constants p and K , we have $|\varphi(t, x, k)| \leq K(1 + |x|^p)$;
9. $C^{1,2}(\bar{Q}_s) = \{\Phi(t, x) \mid \Phi(t, x), \Phi_t, \Phi_x, \Phi_{xx} \text{ are continuous on } \bar{Q}_s\}$;
10. $C_p^{1,2}(\bar{Q}_s) = \{\Phi(t, x) \in C^{1,2}(\bar{Q}_s) \mid \Phi_t, \Phi_x, \Phi_{xx} \text{ satisfy a polynomial growth condition}\}$.

11. We denoted by

$$\mathcal{D}_m \left(\sum_{k=0}^{\infty} v_k(x) p^k \right) = \frac{1}{m!} \frac{\partial^m (\sum_{k=0}^{\infty} u_k(x) p^k)}{\partial p^m} \Big|_{p=0} = u_m,$$

the homotopy derivative, which satisfies the following properties:

$$\mathcal{D}_m \left(\sum_{k=0}^{\infty} v_k(x) p^k \right) = u_m, \quad \mathcal{D}_m \left(p^k \sum_{k=0}^{\infty} u_k(x) p^k \right) = \begin{cases} u_{m-k} & \text{if } 1 \leq k \leq m, \\ 0 & \text{if } k > m. \end{cases}$$

2.2 Problem Formulation

We consider a controlled Markov process $(X(t), \alpha(t))$ defined by the stochastic differential equation,

$$dX(t) = \mu(\alpha(t))dt + b_0 u(t)dt + \sigma(\alpha(t))dW(t), \quad t \in [s, T], \quad (2.1)$$

$$X(s) = x \in [a, b], \quad \alpha(s) = k. \quad (2.2)$$

The control $\underline{u} : \bar{Q}_s \times \mathbb{M} \rightarrow U$ is an $\{\mathcal{F}_t\}_{t \geq s}$ -adapted process on $(\Omega, \mathcal{F}, \mathcal{P})$; for simplicity, we set $u(t) = \underline{u}(t, X(t), \alpha(t))$. If Eq. 2.1 with the initial data $X(s) = x$ has a unique solution $X(\cdot)$, with $u(t) = \underline{u}(t, X(t), \alpha(t))$ belonging to $\mathcal{U}^0(t)$, then we call u an admissible feedback control for initial conditions (s, x, k) and $\pi = (\Omega, \{\mathcal{F}_s\}, \mathcal{P}, X(s), u(s))$. Let

$$T_k(x) = \inf\{t \geq s : X(t) = a, \text{ or } b \mid X(s) = x \in (a, b), \alpha(s) = k\}. \quad (2.3)$$

denote the stopping time, and set $\tau_k(x) = T \wedge T_k(x)$.

For a given control $u \in \mathcal{U}(s, x)$, let

$$C(s, x, u(s), k) = \int_s^{\tau_k(x)} L(t, X(t), u(t), \alpha(t))dt + g(\tau_k(x), X(\tau_k(x)), \alpha(\tau_k(x))) \quad (2.4)$$

be the cost function, where $L : \bar{Q}_s \times \mathcal{U}(s, x) \times \mathbb{M} \rightarrow \mathbb{R}$ represents the running cost and $g : \bar{Q}_0 \times \mathbb{M} \rightarrow \mathbb{R}$ is the terminal cost. Here, the terminal cost is assumed to be a convex function. Let

$$J(s, x, u, k) = \mathbb{E} \left[C(s, x, u, k) \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right] \quad (2.5)$$

be the expected total cost, where $\mathcal{M}_{\alpha(s)}^{(s, X(s))} \equiv \{X(s) = x, \alpha(s) = k\}$.

The goal is to find an optimal control $u^* \in \mathcal{U}(s, x)$ that minimizes the total cost

$$V(s, x, k) = J(s, x, k, u^*) = \inf_{u(s) \in \mathcal{U}(s, x)} \mathbb{E} \left[J(s, x, u, k) \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right] \quad (2.6)$$

for all $(s, x, k) \in [0, T] \times [a, b] \times \mathbb{M}$. Note that the terminal and boundary conditions are

$$V(T, x, k) = g(T, x, k), \quad \forall (x, k) \in [a, b] \times \mathbb{M} \quad (2.7)$$

and

$$V(s, a, k) = g(s, a, k), \quad V(s, b, k) = g(s, b, k), \quad \forall (s, k) \in [0, T] \times \mathbb{M} \quad (2.8)$$

Throughout the paper, we assume the following:

1. $|g(t_1, y, j, u) - g(t_2, z, l, u)| \leq k_0 |y - z|$, the terminal cost convex, $k_0 \in \mathbb{R}_+^*$.
2. We let $L(t, x, k, u) = \frac{1}{2} q_0 u^2$ where q_0 is a positive constant. This situation is often referenced as “LQG homing” (Whittle 1983).

It is clear that $J(s, \cdot, \cdot, \cdot)$ is convex on $\bar{Q}_s \times \mathbb{M}$ and also $V(s, \cdot, \cdot)$ is convex on $[a, b]$.

3 Dynamic Programming

3.1 Optimal Control

Let $V(\cdot, \cdot, \cdot)$ be the value function defined by Eq. 2.6,

$$V(x, s, k) = \inf_{u(s), s \leq t \leq \tau_k(x)} \mathbb{E} \left[(J(X(s), \alpha(s), s)) \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right] \quad (3.1)$$

Lemma 3.1 *The optimality equation for the problem including the plant (2.1), cost function (2.4), and criterion (3.1) satisfies the Hamilton-Jacobi-Bellman (HJB) equation given by:*

$$0 = \inf_{u(s)} \left\{ \frac{1}{2} q_0 u^2(s) + \mathcal{A}^{u(s)} V(s, x, k) \right\} \quad (s, x, k) \in [0, T] \times [a, b] \times \mathbb{M} \quad (3.2)$$

$$V(T, x, k) = g(T, x, k), \quad \forall (x, k) \in [a, b] \times \mathbb{M} \quad (3.3)$$

$$V(s, b, k) = g(s, b, k), \quad \forall (s, k) \in [0, T] \times \mathbb{M}, \quad (3.4)$$

and

$$V(s, a, k) = g(s, a, k), \quad \forall (s, k) \in [0, T] \times \mathbb{M}, \quad (3.5)$$

where

$$\begin{aligned} \mathcal{A}^{u(s)} V(s, x, k) = & \frac{1}{2} \sigma^2(k) \frac{\partial^2 V(s, x, k)}{\partial x^2} + \frac{\partial V(s, x, k)}{\partial s} \\ & + [\mu(k) + b_0 u(s)] \frac{\partial V(s, x, k)}{\partial x} + \sum_{j=1}^m q_{kj} [V(s, x, j) - V(s, x, k)] \end{aligned} \quad (3.6)$$

Proof If we consider the infinitesimal time interval $(s, s + dt)$, $V(s, x, k)$ can be written as

$$\begin{aligned} V(s, x, k) = & \inf_{u(s)} E \left[\int_s^{s+dt} \frac{1}{2} q_0 u^2(t) dt \right. \\ & \left. + \int_{s+dt}^{\tau_k(x)} \frac{1}{2} q_0 u^2(t) dt + g(\tau_k(x), X(\tau_k(x)), \alpha(\tau_k(x))) \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right]. \end{aligned} \quad (3.7)$$

where $\mathcal{M}_{\alpha(s)}^{(s, X(s))} \equiv \{X(s) = x, \alpha(s) = k\}$. The first summand on the right side simplifies to $\mathbb{E} \left[\int_s^{s+dt} \frac{1}{2} q_0 u^2(t) dt \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right] = \frac{1}{2} q_0 u^2(s) dt$. Also,

$$\begin{aligned} & E \left[E \left(\psi_{\tau_k(x)}(s + dt) \mid \mathcal{M}_{\alpha(s+dt)}^{(s, X(s+dt))}, \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right) \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right] \\ & = E \left[\psi_{\tau_k(x)}(s + dt) \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right], \end{aligned}$$

where $\psi_{\tau_k(x)}(s + dt) = \int_{s+dt}^{\tau_k(x)} \frac{1}{2} q_0 u^2(t) dt + g(\tau_k(x), X(\tau_k(x)), \alpha(\tau_k(x)))$.

By making use of Bellman's principle of optimality we obtain:

$$\begin{aligned} V(s, x, k) = & \inf_{u(s)} E \left[\frac{1}{2} q_0 u^2(s) dt \right. \\ & \left. + V(s + dt, X(s + dt), \alpha(s + dt)) \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right]. \end{aligned} \quad (3.8)$$

By using the following hypothesis

1. Plant equation (2.1)

$$\begin{aligned} X(s+dt) &= X(s) + [\mu(k) + b_0 u(t)]dt + \sigma(k)\Delta W(t) \\ &= X(s) + f(W, k, u, t), \end{aligned} \quad (3.9)$$

where

$$f(W, k, u, t) = [\mu(k) + b_0 u(t)]dt + \sigma(k)\Delta W(t). \quad (3.10)$$

2. Taylor expansion of

$$\begin{aligned} V(s+dt, X(s+dt), \alpha(s+dt)) &= V(s, X(s), \alpha(s+dt)) \\ &+ \frac{\partial V(s, X(s), \alpha(s+dt))}{\partial s} dt \\ &+ \frac{\partial V(s, X(s), \alpha(s+dt))}{\partial x} f(W, k, u, s) \\ &+ \frac{1}{2} \frac{\partial^2 V(s, X(s), \alpha(s+dt))}{\partial x^2} f^2(W, k, u, s). \end{aligned} \quad (3.11)$$

3. linearity of the conditional expectation

$$\begin{aligned} &E \left[V(s, X(s), \alpha(s+dt) = j) \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right] \\ &= \sum_{\substack{j=1 \\ j \neq k}}^m V(s, X(s), j) q_{kj} dt + V(s, X(s), k)(1 + q_{kk} dt), \end{aligned} \quad (3.12)$$

$$\begin{aligned} &E \left[\frac{\partial V(s, X(s), \alpha(s+dt) = j)}{\partial s} \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right] \\ &= \sum_{\substack{j=1 \\ j \neq k}}^m \frac{\partial V(s, X(s), j)}{\partial s} q_{kj} dt + \frac{\partial V(s, X(s), k)}{\partial s} (1 + q_{kk} dt), \end{aligned} \quad (3.13)$$

$$\begin{aligned} &E \left[\frac{\partial V(s, X(s), \alpha(s+dt) = j)}{\partial x} \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right] \\ &= \sum_{\substack{j=1 \\ j \neq k}}^m \frac{\partial V(s, X(s), j)}{\partial x} q_{kj} dt + \frac{\partial V(s, X(s), k)}{\partial x} (1 + q_{kk} dt), \end{aligned} \quad (3.14)$$

$$\begin{aligned} &E \left[\frac{\partial^2 V(s, X(s), \alpha(s+dt) = j)}{\partial x^2} \mid \mathcal{M}_{\alpha(s)}^{(s, X(s))} \right] \\ &= \sum_{\substack{j=1 \\ j \neq k}}^m \frac{\partial^2 V(s, X(s), j)}{\partial x^2} q_{kj} dt + \frac{\partial^2 V(s, X(s), k)}{\partial x^2} (1 + q_{kk} dt), \end{aligned} \quad (3.15)$$

4. $E[W(s+dt)] = 0$, $E[W^2(s+dt)] = \sigma^2(k)dt$, the independence between α and W .

we obtain

$$\begin{aligned} V(s, x, k) = & V(s, x, k) + \inf_{u(s)} \left[\frac{1}{2} q_0 u^2(s) + \frac{1}{2} \sigma^2(k) \frac{\partial^2 V(s, x, k)}{\partial x^2} \right. \\ & + [\mu(k) + b_0 u(s)] \frac{\partial V(s, x, k)}{\partial x} + \frac{\partial V(s, x, k)}{\partial s} \\ & \left. + \sum_{j=1}^m q_{kj} [V(s, x, j) - V(s, x, k)] + O(dt) \right] dt. \end{aligned} \quad (3.16)$$

□

3.2 Verification Theorem

The following verification theorem is an adaptation of Theorem 8.1 by Fleming and Soner (2006) in the LQG with switching case.

Proposition 3.2 (Verification Theorem) *Suppose that there exists a function $\varphi : \bar{Q}_s \times \mathbb{M} \rightarrow \mathbb{R}$ such that: $\varphi(\cdot, \cdot, l) \in C^{1,2}(\bar{Q}_s) \cap C_p^{1,2}(\bar{Q}_s)$ for each $l \in \mathbb{M}$, and φ satisfies the HJB equation, (3.2)–(3.5). Then:*

1. *For any initial condition $(s, x, \alpha(\cdot)) \in [0, T) \times [a, b] \times \mathbb{M}$, and any admissible feedback control $u(\cdot)$,*

$$\varphi(s, x, k) \leq J(s, x, k, u(\cdot)). \quad (3.17)$$

2. *Moreover, if $u^*(\cdot)$ is an admissible feedback control such that*

$$u^*(s) = \operatorname{argmin}_{u(s) \in \mathcal{U}(s, x)} \left[\mathcal{A}^{u(s)} \varphi(s, x(s), k) + L(s, x(s), k, u) \right], \quad (3.18)$$

then

$$\varphi(s, x, k) = V(s, x, k) = J(s, x, u^*, k) \quad \forall (s, x, k) \in (s, T) \times [a, b] \times \mathbb{M}, \quad (3.19)$$

and $u^*(\cdot)$ is an optimal control.

3.3 Uncontrolled System

Differentiating $\mathcal{A}^{u(s)} \varphi(s, x(s), k) + L(s, x(s), k, u)$ with respect to $u(s)$ yields the optimal control $u^*(s)$ of $u(s)$, where

$$u^*(s) = -\frac{b_0}{q_0} \frac{\partial V(s, x, k)}{\partial x}. \quad (3.20)$$

Substituting this value into Eq. 3.2 results in two coupled, second-order, non-linear equations,

$$\begin{aligned} \frac{1}{2} \sigma^2(k) \frac{\partial^2 V(s, x, k)}{\partial x^2} - \frac{b_0^2}{2q_0} \left(\frac{\partial V(s, x, k)}{\partial x} \right)^2 + \mu(k) \frac{\partial V(s, x, k)}{\partial x} \\ + \frac{\partial V(s, x, k)}{\partial s} + \sum_{j=1}^m q_{kj} [V(s, x, j) - V(s, x, k)] = 0. \end{aligned} \quad (3.21)$$

With Eq. 3.21, we may now approximate the value function for the uncontrolled system via OHAM (Liao and Zhao 2016). The accuracy of the analytical approximation with respect to the sum truncation order is compared to a numerical approximation.

In the sequel we assume $s = 0$, $\sigma(k) \neq 0$ and $m = 2$; we let $V(x, k) = V_k(x)$, we set $R_k = -\frac{1}{\sigma^2(k)} \frac{b_0^2}{q_0}$, $S_k = \frac{2}{\sigma^2(k)} \mu(i)$, $T_k = \frac{2}{\sigma^2(k)}$.

4 Computational Algorithm

4.1 Optimal Homotopy Analysis Method OHAM

In order to obtain an analytical approximation to the value function $V_i(x)$ via OHAM, we first begin with the following equation

$$\begin{aligned} \mathcal{N}_i[V_i(x)] &= V_i(x)'' + R_i(V_i(x)')^2 + S_i V_i(x)' \\ &\quad + T_i q_{ii} [V_i(x) - V_i(x)] = 0, \end{aligned} \quad (4.1)$$

subject to the boundary conditions

$$V_i(a) = \alpha_i, \quad V_i(b) = \beta_i, \quad (4.2)$$

Let rewrite Eq. 4.1 as follow:

$$\mathcal{N}_i[V_i(x)] = V_i(x)'' + r V_i(x)' + R_i(V_i(x)')^2 + (S_i - r) V_i(x)' + T_i q_{ii} [V_i(x) - V_i(x)]$$

with an r optimization parameter that will be obtained by minimizing the errors on the value function. By using the technique of OHAM (Liao and Zhao 2016), we construct the so-called zeroth-order deformation equation.

$$(1 - p) \mathcal{L}_i[V_i(x; p) - v_{i,0}(x)] = p \hbar_i \mathcal{H}_i(x) \mathcal{N}_i[V_i(x; p)], \quad (4.3)$$

where $p \in [0, 1]$ is the embedding parameter, $\hbar_i \neq 0$ is an auxiliary parameter (convergence controller), and $\mathcal{L}_i$ are auxiliary linear operators. The initial guess is $V_i(x; 0) = v_{i,0}(x)$ and $\mathcal{H}_i(x)$ denote the nonzero auxiliary function. We see when $p = 0$ and $p = 1$, $V_i(x; p) = v_{i,0}(x)$ and $V_i(x; 1) = V_i(x)$, which must be one of the solutions to the nonlinear equation $\mathcal{N}_i[V_i(x)] = 0$, $i = 1, 2$, as proven by Liao (1995, 2004).

Expanding $V_i(x; p)$ in via a Taylor series with respect to p , one gets

$$V_i(x; p) = v_{i,0}(x) + \sum_{m=1}^{\infty} v_{i,m}(x) p^m, \quad \text{where } v_{i,m}(x) = \mathcal{D}_m(V_i(x; p)). \quad (4.4)$$

Applying the m th-order homotopy-derivative operator (2.1) to both sides of the zeroth-order deformation equations (4.3), it is straightforward to obtain the m th-order deformation equation:

$$\mathcal{L}_i[v_{i,m}(x) - \chi_{i,m} v_{i,m-1}(x)] = \hbar_i \mathcal{H}_i(x) \mathfrak{R}_{i,m-1}(x), \quad (4.5)$$

where

$$\begin{aligned} \mathfrak{R}_{i,m-1}(x) &= \mathcal{D}_m(p \mathcal{N}_i(V_i(x; p))) \\ &= \frac{\partial^2 v_{i,m-1}(x)}{\partial x^2} + R_i \sum_{j=0}^{m-1} \left(\frac{\partial v_{i,j}(x)}{\partial x} \frac{\partial v_{i,m-1-j}(x)}{\partial x} \right) + S_i \frac{\partial v_{i,m-1}(x)}{\partial x} \\ &\quad + T_i q_{ii} [v_{i,m-1}(x) - v_{i,m-1}(x)], \end{aligned} \quad (4.6)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1. \end{cases}$$

Knowing that we have much freedom in choosing our initial guess (Liao 2003), we choose our initial guess in terms of exponential functions:

$$v_{i,0}(x) = \sum_{k=0}^1 a_{ik} e^{-krx} \quad \text{with } a_{i0} = \alpha_i - \frac{\alpha_i - \beta_i}{e^{-ra} - e^{-rb}} e^{-ra}, \quad a_{i1} = \frac{\alpha_i - \beta_i}{e^{-ra} - e^{-rb}}.$$

From our initial guess, we can choose a finite base functions and auxiliary function as follow:

$$\begin{aligned} \mathcal{B}_n(x) &= \{e^{-nrx} \mid n \geq 0, r > 0\}, & \hat{\mathcal{B}}_n(x) &= \{e^{-nrx} \mid n \geq 2, r > 0\} \\ \mathcal{B}_n^*(x) &= \{e^{-nrx} \mid n = 0, n = 1, r > 0\}, \\ \mathcal{H}_i(x) &= e^{-2rx} \end{aligned} \quad (4.7)$$

Let

$$V = \text{span}\{\mathcal{B}_n(x)\}, \quad V^* = \text{span}\{\mathcal{B}_n^*(x)\}, \quad \hat{V} = \text{span}\{\hat{\mathcal{B}}_n(x)\}.$$

Now we are going to define explicitly the linear operator and its inverse as follows:

$$\begin{aligned} \mathcal{L}_i : \hat{V} &\longrightarrow \hat{V} \\ \phi &\mapsto \mathcal{L}_i(\phi) = \frac{d^2\phi}{dx^2} + r \frac{d\phi}{dx} \\ \mathcal{L}_i^{-1} : \hat{V} &\longrightarrow \hat{V} \\ e^{-nrx} &\mapsto \mathcal{L}_i^{-1}(e^{-nrx}) = \frac{e^{-nrx}}{r^2(n^2 - n)} \end{aligned}$$

Remark 4.1 The method of directly defining inverse mapping (MDDiM) (Liao and Zhao 2016) we can define the inverse mapping $\mathcal{L}_i^{-1}$, without calculating any inverse operators i.e $\mathcal{L}_i$ does not need to be specified.

Since $\lim_{n \rightarrow +\infty} e^{-nrx}$, the base solution $\mathcal{B}_n(x)$ is finite.

The following proposition is an adaptation of the convergence-theorem (Liao and Zhao 2016) in the case of a nonlinear coupled differential system.

Proposition 4.2 *If the convergence-control parameters $\hbar_i, r$ are properly chosen so that the series*

$$V_i(x) = v_{i,0}(x) + \sum_{k=1}^{\infty} v_{i,k}(x), \quad i = 1, 2, \quad (4.8)$$

is absolutely convergent, then it must be a solution of the original equation (4.1–4.2).

The parameter r can be determined by minimizing $\mathcal{N}_i(V_i(x, r))$.

Let $\bar{V}_i(x, r) = \sum_{k=0}^3 v_{ik}(x, r)$; if $\mathcal{N}_i[\bar{V}_i(x, r)] = 0$, then $\bar{V}_i(x, r)$ is the exact solution. If $\mathcal{N}_i[\bar{V}_i(x, r)] \neq 0$, then there are residual error functions that can be evaluated at any point x in the domain of the problem. Taking the affine combination square of the L^2 -norm of error functions

$$\mathcal{E}(x, r) = \int_a^b \sum_{i=1}^2 \{\mathcal{N}_i[\bar{V}_i(x, r)]\}^2 dx,$$

we obtain

$$\bar{r} = \underset{r}{\operatorname{argmin}} \mathcal{E}(x, r)$$

4.2 Finite difference Method for the Uncontrolled Equation

The differential-difference equation, Eq. 3.21, in the interval $[a, b]$ is discretized. We let $h = \frac{b-a}{N+1}$, $x_i = x_0 + ih$, $i = 0, 1, 2, \dots, N+1$, $x_0 = a$, and $x_{N+1} = b$. We use the central difference approximation for the second-order derivative,

$$\frac{\sigma^2(j)}{2} \frac{d^2 V_j}{dx^2} \approx \frac{\sigma^2(j)}{2} \frac{V_{j,i+1} - 2V_{j,i} + V_{j,i-1}}{h^2}, \quad (4.9)$$

where $j = 1, 2$ denotes the regime. The first-order derivative in the advection-like term is evaluated using the first-order upwind scheme criterion,

$$\mu(j) \frac{dV_j}{dx} \approx \Theta(\mu(j)) \left(\frac{V_{j,i} - V_{j,i-1}}{h} \right) + \Theta(-\mu(j)) \left(\frac{V_{j,i+1} - V_{j,i}}{h} \right), \quad (4.10)$$

where Θ is the Heaviside step function. The first-order derivative that is squared is approximated with a central difference,

$$\frac{b_0^2}{2q_0} \left(\frac{dV_j}{dx} \right)^2 = \frac{b_0^2}{8q_0 h^2} (V_{j,i+1} - V_{j,i-1})^2. \quad (4.11)$$

The discretization of Eq. 3.21 follows as

$$\begin{aligned} & \frac{\sigma^2(j)}{2} \frac{V_{j,i+1} - 2V_{j,i} + V_{j,i-1}}{h^2} - \frac{b_0^2}{8q_0 h^2} (V_{j,i+1} - V_{j,i-1})^2 \\ & + \Theta(\mu(j)) \left(\frac{V_{j,i} - V_{j,i-1}}{h} \right) + \Theta(-\mu(j)) \left(\frac{V_{j,i+1} - V_{j,i}}{h} \right) \\ & + \sum_{l=1}^2 q_{jl} (V_{l,i} - V_{j,i}) = 0. \end{aligned} \quad (4.12)$$

By letting $V_{j,0} = \alpha_j$ and $V_{j,N+1} = \beta_j$, we obtain the following matrix equation,

$$A(j)V(j) = F(j). \quad (4.13)$$

with

$$A(j) = \begin{bmatrix} d(j) & e(j) & 0 & \dots & \dots & \dots & \dots & 0 \\ c(j) & d(j) & e(j) & 0 & \dots & \dots & \dots & 0 \\ 0 & c(j) & d(j) & e(j) & 0 & \dots & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & e(j) \\ 0 & 0 & 0 & 0 & 0 & 0 & c(j) & d(j) \end{bmatrix}.$$

The coefficients of the tridiagonal matrix are

$$c(j) = \frac{\sigma^2(j)}{2h^2} + \frac{\mu(j)}{h} \Theta(-\mu(j)), \quad (4.14)$$

$$d(j) = \frac{\mu(j)}{h} \Theta(\mu(j)) - \frac{\sigma^2(j)}{h^2} - \frac{\mu(j)}{h} \Theta(-\mu(j)), \quad (4.15)$$

$$e(j) = \frac{\sigma^2(j)}{2h^2} - \frac{\mu(j)}{h} \Theta(\mu(j)). \quad (4.16)$$

The F vector is given by

$$F(j) = (F_{j,1}, \dots, F_{j,N})^T$$

$$F_{j,i} = \frac{b_0^2}{8q_0h^2} (V_{j,i+1} - V_{j,i-1})^2 - \sum_{l=1}^2 q_{jl} (V_{l,i} - V_{j,i}) - H_{ji} \quad (4.17)$$

where

$$H_{ji} = \begin{cases} c(j) \alpha_j & \text{if } i = 1 \\ e(j) \beta_j & \text{if } i = N \\ 0 & \text{elsewhere.} \end{cases}$$

$V(j) = (V_{j,1}, \dots, V_{j,N})^T$ are obtained iteratively. Taking an initial approximation $V^{(0)}(j) = (V_{j,1}^{(0)}, \dots, V_{j,N}^{(0)})^T$, we solve

$$V(j)^{(k+1)} = A^{-1}(j)F^{(k)}(j).$$

for $k = 1, 2, 3, \dots$ until the difference between the k th and $(k - 1)$ th vectors is negligible. The numerical approximation was computed using Python with NumPy. The Heaviside step function was defined from the SymPy library.

Remark 4.3 We can discretize the uncontrolled system and use the finite-difference method to get both the value function and optimal control (Wang and Forsyth 2008). In this paper we focus on the uncontrolled system with the LQG control defined with respect to the uncontrolled value function. Thus, we are also able to get an approximate analytical expression via OHAM to compare with the numerical results.

5 Extending an Application to Include Regime Switching

The Gompertz law is known to describe the growth of tumors in patients (Bassukas 1994). The size of a tumor, $x(t)$, is modeled as

$$\frac{dx}{dt} = A_1x + A_2x \ln x, \quad (5.1)$$

where A_1 is the tumor's growth rate and A_2 is the control rate. If the growth rate varies over time, $\theta(t) = A_1 + \sigma \varepsilon(t)$, with A_1 being the constant mean value, $\theta(t)$, $\sigma > 0$ being the diffusion coefficient, and $\varepsilon(t)$ being a normal distributed white noise, then

$$dx = \{A_1x + A_2x \ln x\}dt + \sigma x dW_t. \quad (5.2)$$

The standard Wiener process is denoted by dW_t in Eq. 5.2. The exponent, $\psi = -\ln x$, follows the Ornstein-Uhlenbeck process as a consequence of Eq. 5.1,

$$d\psi = \left\{ \left(\frac{1}{2}\sigma^2 - A_1 \right) + A_2\psi \right\} dt + \sigma dW_t. \quad (5.3)$$

Let us assume that $\sigma(k)$ depends on some state of the Markov chain; then, we may transform (5.3) into a controlled process by setting

$$u(t) = \frac{A_2(k)}{b_0} \psi(t) \quad (5.4)$$

and

$$\mu(k) = \frac{1}{2}\sigma^2(k) - A_1(k), \quad (5.5)$$

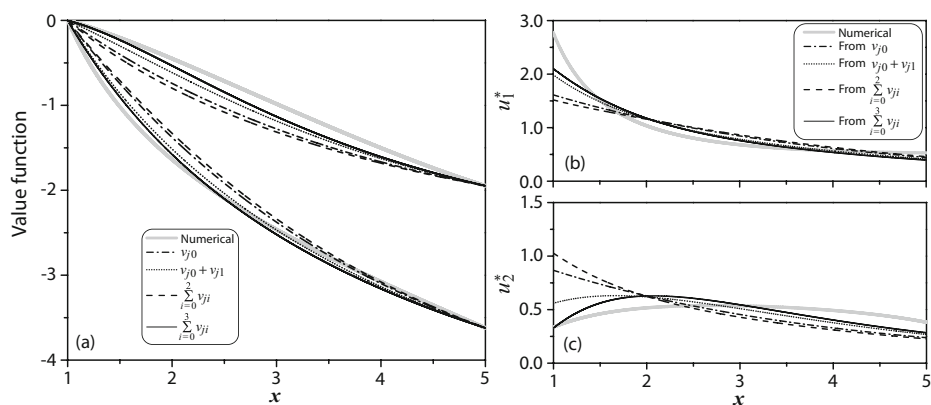


Fig. 1 **a** The value function for both regimes for the parameters $\bar{h}_1 = \bar{h}_2 = -1$, $q_0 = 1$, $q_{12} = q_{21} = 0.2$, $b_0 = 1$, $\sigma(1) = 1.5$, $\sigma(2) = 1.1$, $\mu(1) = 0.4$, $\mu(2) = -0.8$, and third-order optimized $r = 0.322943$. The third-order perturbation from OHAM (solid black line) shows a trending convergence to the numerical approximation (thick gray line) relative to the initial guess (dash-dot line). Approximations with odd-ordered truncations appear to be closer with the same optimization parameter r relative to even ordered truncations using the odd-ordered optimization parameter. The optimum control determined from the value function is shown in **(b)** and **(c)**

where $u(t)$ is the control drug concentration at time t . We substitute (5.4) and (5.5) into (5.3) to get

$$d\psi(t) = \mu(k)dt + b_0 u(t)dt + \sigma(k)dW_t, \quad \psi(t_0) = \psi_0. \quad (5.6)$$

According to Section 3.3, we found the minimal value of the control drug given by

$$u^* = -\frac{b_0}{q_0} \frac{\partial V(\psi, k)}{\partial \psi}$$

and $V(\psi, k)$ is approximately equal to $\bar{V}(\psi, k)$.

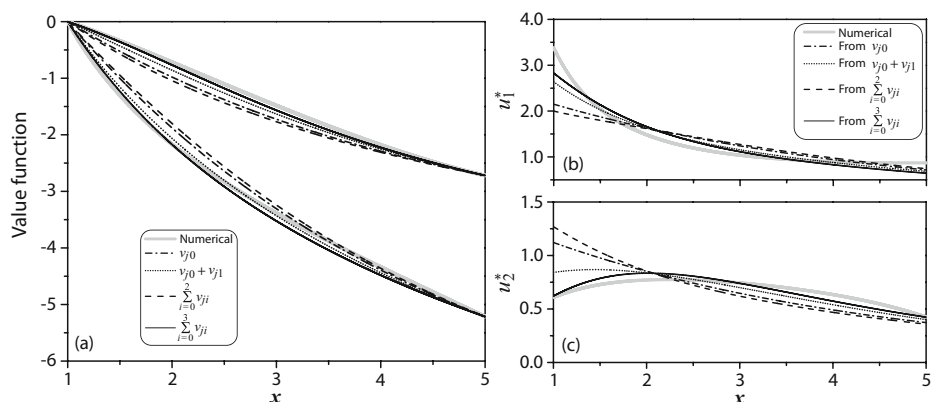


Fig. 2 **a** The value function for both regimes for the parameters $\bar{h}_1 = \bar{h}_2 = -1$, $q_0 = 1$, $q_{12} = 0.3$, $q_{21} = 0.2$, $b_0 = 1$, $\sigma(1) = 1.8$, $\sigma(2) = 1.3$, $\mu(1) = 0.4$, $\mu(2) = -0.8$, and third-order optimized $r = 0.275079$. The optimum control determined from the value function is shown in **(b)** and **(c)**

Suppose the controlled cancer growth can be modeled using some generic parameters. The boundaries are given by $V_i(a) = \alpha_i$ and $V_i(b) = \beta_i$, where we have set $a = 1$ and a relatively long-time boundary $b = 5$. Figure 1 shows the value function for first initial guess and following three iterations in the OHAM approximation. The converged successive iteration approximation using finite differences is also shown in Fig. 1. The value function approximations from higher-order OHAM perturbations begin to converge towards the numerical approximation. Figure 1b and c are the optimal control values calculated using (3.20). The numerical solution is very precise. There is a large difference between u_j^* from low-order OHAM approximations with the third-order optimized r and the numerical solutions; however, we see a definite improvement in the OHAM approximation for approximations truncated to first- and third-order. The numerical and OHAM approximations to a second set of generic parameters is shown in Fig. 2.

6 Conclusion

We have extended the results proved by Whittle (1983) and Lefebvre (2014) to the case of a one-dimensional Markov regime switching model. For simplicity, we focused on the case where the underlying Markov chain has two states. We showed approximate solutions to the Gompertz law via an analytical expression using OHAM as well as a successive iteration method based on finite differences. An initial guess of a function constructed from exponentials was used in the OHAM approach, which only required a single optimization parameter for the arbitrarily gained inverse linear operator. The results of this paper combined with the OHAM and/or successive iteration method can be followed to solve many more problems in mathematics and sciences (Dawson and Kounta 2019).

The very nature of a linear quadratic Gaussian allows us to determine the control from the value function determined by the uncontrolled problem. The presented method can be directly applied to population growth of bacteria or tumor growth with parameters rooted in environmental factors that include regime switching. This method can possibly be further extended to include a Markov chain with more than two states as well as extending the results to include jump-diffusion processes. The resultant analytical expressions obtained from OHAM can approximate the value function for systems described by regime switching stochastic equations, although higher-order expression may be needed for increased precision for large domains. Complicated nonlinear functions will likely result in lengthy expressions that require computer algebra programs to solve and store the expressions.

Acknowledgements MK would like to thank the University of The Bahamas Internal Grants programme for Research, Creative and Artistic Proposals for supporting this research project. NJD would like to thank the Hawaii Pacific University, College of Natural and Computational Sciences, Scholarly Endeavors Program for supporting this research project.

Appendix A: Proof of Proposition 4.2

Proof

$$\begin{aligned}\mathcal{L}[v_{i,m}(x)] &= \mathcal{L} \left\{ \chi_{i,m} v_{i,m-1}(x) + \hbar_i \mathcal{L}_i^{-1} \left[e^{-2rx} \mathfrak{H}_{i,m-1}(x) \right] + c_{i0}^m + c_{i1}^m e^{-rx} \right\} \\ &= \chi_{i,m} \mathcal{L}_i[v_{i,m-1}(x)] + \hbar_i e^{-2rx} \mathfrak{H}_{i,m-1}(x)\end{aligned}\quad (1)$$

Because $\mathcal{L} \circ \mathcal{L}^{-1}(x) = x$, $\forall x \in \hat{V}$ and $\mathcal{L}(x) = 0$, $\forall x \in V^*$, (1) implies

$$\begin{aligned} \sum_{j=1}^m (\mathcal{L}[v_{i,j}(x)] - \mathcal{L}[v_{i,j-1}(x)]) &= \sum_{j=1}^m \left(\hbar_i e^{-2rx} \mathfrak{R}_{i,j-1}(x) \right) \\ \Rightarrow \mathcal{L}[v_{i,m}(x)] &= \mathcal{L}[v_{i,0}(x)] + \sum_{j=1}^m \left(\hbar_i e^{-2rx} \mathfrak{R}_{i,j-1}(x) \right) \end{aligned}$$

Because $\mathcal{L}[v_{i,0}(x)] = 0$, we obtain

$$\lim_{m \rightarrow \infty} \mathcal{L}_i(v_{i,m}(x)) = \hbar_i e^{-2rx} \sum_{j=0}^{\infty} \mathfrak{R}_{i,j}(v_{i,j}(x))$$

Also, Eq. 4.8 is absolutely convergent, and therefore

$$\lim_{m \rightarrow \infty} v_{i,m}(x) = 0.$$

Then

$$\hbar_i e^{-2rx} \sum_{j=0}^{\infty} \mathfrak{R}_{i,j}(x) = \lim_{m \rightarrow \infty} \mathcal{L}_i(v_{i,m}(x)) = \mathcal{L}_i\left(\lim_{m \rightarrow \infty} v_{i,m}(x)\right) = \mathcal{L}_i(0) = 0$$

Because $\hbar_i \neq 0$ and $e^{-2rx} \neq 0$, it follows that

$$\sum_{j=0}^{\infty} \mathfrak{R}_{i,j}(x) = 0.$$

Also note that the Taylor series of

$$\mathcal{N}_i \left[\sum_{j=0}^{\infty} v_{ij}(x) p^j \right] = \sum_{j=0}^{\infty} \mathfrak{R}_{i,j}(x) p^j,$$

at $p = 1$

$$\mathcal{N}_i \left[\sum_{j=0}^{\infty} v_{ij}(x) \right] = \sum_{j=0}^{\infty} \mathfrak{R}_{i,j}(x) = 0.$$

□

Appendix B: Recursive Calculation of the Value Function Via OHAM

By using Eq. 4.5, we obtain

$$v_{i,m}(x) = \chi_{i,m} v_{i,m-1}(x) + \hbar_i \mathcal{L}_i^{-1} \left[e^{-2rx} \mathfrak{R}_{i,m-1}(x) \right] + c_{i0}^m + c_{i1}^m e^{-rx},$$

where $\mathcal{L}_i(c_{i0}^m + c_{i1}^m e^{-rx}) = 0$ and c_{i0}^m, c_{i1}^m are constants of integration, which will be determined from the boundary conditions. By making use of Eq. 4.6, we obtain

$$\mathfrak{R}_{i,0}(x) = R_i r^2 a_{i1}^2 e^{-2rx} + [a_{i1} r^2 - S_i a_{i1} r + T_i q_{i\bar{i}}(a_{i\bar{i}} - a_{i1})] e^{-rx} + T_i q_{i\bar{i}}(a_{i\bar{i}} - a_{i0})$$

and

$$v_{i,1}(x) = \hbar_i R_i a_{i1}^2 \frac{e^{-4rx}}{12} + \hbar_i [a_{i1}r^2 - S_i a_{i1}r + T_i q_{i\bar{i}}(a_{i\bar{i}} - a_{i1})] \frac{e^{-3rx}}{6r^2} \\ + \hbar_i T_i q_{i\bar{i}}(a_{i\bar{i}0} - a_{i0}) \frac{e^{-2rx}}{2r^2} + c_{i0}^1 + c_{i1}^1 e^{-rx}$$

By letting,

$$\alpha_{i4} = \frac{\hbar_i R_i a_{i1}^2}{12}, \quad \alpha_{i3} = \frac{\hbar_i [a_{i1} - S_i a_{i1} + T_i q_{i\bar{i}}(a_{i\bar{i}} - a_{i1})]}{6r^2}, \quad \alpha_{i2} = \frac{\hbar_i T_i q_{i\bar{i}}(a_{i\bar{i}0} - a_{i0})}{2r^2},$$

we obtain

$$v_{i1}(x) = \sum_{k=2}^4 \alpha_{ik} e^{-krx} + c_{i0}^1 + c_{i1}^1 e^{-rx}$$

with

$$c_{i1}^1 = \frac{\sum_{k=2}^4 \alpha_{ik} e^{-kra} - \sum_{k=2}^4 \alpha_{ik} e^{-krb}}{e^{-rb} - e^{-ra}}, \quad c_{i0}^1 = -\sum_{k=2}^4 \alpha_{ik} e^{-kra} - c_{i1}^1 e^{-ra}$$

The second iteration follows as

$$\mathfrak{N}_{i,1}(x) = 8r^2 R_i a_{i1} \alpha_{i4} e^{-5rx} + \left[16r^2 \alpha_{i4} + 6R_i r^2 a_{i1} \alpha_{i3} - 4S_i r \alpha_{i4} + T_i q_{i\bar{i}}(\alpha_{i\bar{4}} - \alpha_{i4}) \right] e^{-4rx} \\ + \left[9\alpha_{i3}r^2 + 4R_i r^2 a_{i1} \alpha_{i2} - 3S_i r \alpha_{i3} + T_i q_{i\bar{i}}(\alpha_{i\bar{3}} - \alpha_{i3}) \right] e^{-3rx} \\ + \left[4r^2 \alpha_{i2} + 2R_i r^2 a_{i1} c_{i1}^1 - 2S_i r \alpha_{i2} + T_i q_{i\bar{i}}(\alpha_{i\bar{2}} - \alpha_{i2}) \right] e^{-2rx} \\ + \left[c_{i1}^1 r^2 - S_i r c_{i1}^1 + T_i q_{i\bar{i}}(c_{i\bar{1}}^1 - c_{i1}^1) \right] e^{-rx} + T_i q_{i\bar{i}}(c_{i\bar{0}}^1 - c_{i0}^1),$$

which, in turn, gives

$$v_{i,2}(x) = v_{i,1}(x) + 8\hbar_i R_i a_{i1} \alpha_{i4} \frac{e^{-7rx}}{42} \\ + \hbar_i \left[16r^2 \alpha_{i4} + 6R_i r^2 a_{i1} \alpha_{i3} - 4S_i r \alpha_{i4} + T_i q_{i\bar{i}}(\alpha_{i\bar{4}} - \alpha_{i4}) \right] \frac{e^{-6rx}}{30r^2} \\ + \hbar_i \left[9\alpha_{i3}r^2 + 4R_i r^2 a_{i1} \alpha_{i2} - 3S_i r \alpha_{i3} + T_i q_{i\bar{i}}(\alpha_{i\bar{3}} - \alpha_{i3}) \right] \frac{e^{-5rx}}{20r^2} \\ + \hbar_i \left[4r^2 \alpha_{i2} + 2R_i r^2 a_{i1} c_{i1}^1 - 2S_i r \alpha_{i2} + T_i q_{i\bar{i}}(\alpha_{i\bar{2}} - \alpha_{i2}) \right] \frac{e^{-4rx}}{12r^2} \\ + \hbar_i \left[c_{i1}^1 r^2 - S_i r c_{i1}^1 + T_i q_{i\bar{i}}(c_{i\bar{1}}^1 - c_{i1}^1) \right] \frac{e^{-3rx}}{6r^2} + \hbar_i T_i q_{i\bar{i}}(c_{i\bar{0}}^1 - c_{i0}^1) \frac{e^{-2rx}}{2r^2} \\ + c_{i0}^2 + c_{i1}^2 e^{-rx}.$$

By letting,

$$\begin{aligned}\beta_{i7} &= \frac{8\hbar_i R_i a_{i1} \alpha_{i4}}{42}, \quad \beta_{i6} = \frac{\hbar_i [16r^2 \alpha_{i4} + 6R_i r^2 a_{i1} \alpha_{i3} - 4S_i r \alpha_{i4} + T_i q_{i\bar{i}}(\alpha_{i4} - \alpha_{i4})]}{30r^2}, \\ \beta_{i5} &= \frac{\hbar_i [9\alpha_{i3} r^2 + 4R_i r^2 a_{i1} \alpha_{i2} - 3S_i r \alpha_{i3} + T_i q_{i\bar{i}}(\alpha_{i3} - \alpha_{i3})]}{20r^2} \\ \beta_{i4} &= \frac{\hbar_i [4r^2 \alpha_{i2} + 2R_i r^2 a_{i1} c_{i1}^1 - 2S_i r \alpha_{i2} + T_i q_{i\bar{i}}(\alpha_{i2} - \alpha_{i2})]}{12r^2} + \alpha_{i4} \\ \beta_{i3} &= \frac{\hbar_i [c_{i1}^1 r^2 - S_i r c_{i1}^1 + T_i q_{i\bar{i}}(c_{i1}^1 - c_{i1}^1)]}{6r^2} + \alpha_{i3}, \quad \beta_{i2} = \frac{\hbar_i T_i q_{i\bar{i}}(c_{i0}^1 - c_{i0}^1)}{2r^2} + \alpha_{i2},\end{aligned}$$

we obtain

$$v_{i,2}(x) = \sum_{k=2}^7 \beta_{ik} e^{-krx} + c_{i0}^1 + c_{i1}^1 e^{-rx} + c_{i0}^2 + c_{i1}^2 e^{-rx},$$

with

$$\begin{aligned}c_{i1}^2 &= \frac{\sum_{k=2}^7 \beta_{ik} e^{-kra} - \sum_{k=2}^7 \beta_{ik} e^{-krb}}{e^{-rb} - e^{-ra}} - c_{i1}^1, \\ c_{i0}^2 &= -\sum_{k=2}^7 \beta_{ik} e^{-kra} - c_{i1}^2 e^{-ra} - c_{i0}^1 - c_{i1}^1 e^{-ra}.\end{aligned}$$

The third iteration follows as

$$\begin{aligned}\Re_{i,2}(x) &= 49r^2 R_i \beta_{i7}^2 e^{-14rx} + 84r^2 \beta_{i6} \beta_{i7} R_i e^{-13rx} + (70\beta_{i5} \beta_{i7} + 36\beta_{i6}^2) R_i r^2 e^{-12rx} \\ &+ (56\beta_{i4} \beta_{i7} + 60\beta_{i6} \beta_{i5}) r^2 R_i e^{-11rx} + (42\beta_{i3} \beta_{i7} + 48\beta_{i4} \beta_{i6} + 25\beta_{i5}^2) r^2 R_i e^{-10rx} \\ &+ (28\beta_{i2} \beta_{i7} + 36\beta_{i3} \beta_{i6} + 40\beta_{i4} \beta_{i5}) r^2 R_i e^{-9rx} + [(14\beta_{i7}(c_{i1}^2 + c_{i1}^1) + 24\beta_{i2} \beta_{i6} \\ &+ 30\beta_{i5} \beta_{i3} + 16\beta_{i4}^2) + 14a_{i1} \beta_{i7}] r^2 R_i e^{-8rx} \\ &+ [(12(c_{i1}^1 + c_{i1}^2) \beta_{i6} + 20\beta_{i5} \beta_{i2} + 24\beta_{i3} \beta_{i4} + 12a_{i1} \beta_{i6}) r^2 R_i + 49r^2 \beta_{i7} \\ &- 7S_i r \beta_{i7} + T_i q_{i\bar{i}}(\beta_{i7} - \beta_{i7})] e^{-7rx} \\ &+ [(10(c_{i1}^1 + c_{i1}^2) \beta_{i5} + 16\beta_{i4} \beta_{i2} + 9\beta_{i3}^2 + 10a_{i1} \beta_{i5}) r^2 R_i + 36r^2 \beta_{i6} - 6S_i r \beta_{i6} \\ &+ T_i q_{i\bar{i}}(\beta_{i6} - \beta_{i6})] e^{-6rx} \\ &+ [(8(c_{i1}^1 + c_{i1}^2) \beta_{i4} + 12\beta_{i3} \beta_{i2} + 8a_{i1} \beta_{i4}) r^2 R_i + 25r^2 \beta_{i5} - 5S_i r \beta_{i5} \\ &+ T_i q_{i\bar{i}}(\beta_{i5} - \beta_{i5})] e^{-5rx} \\ &+ [(6(c_{i1}^1 + c_{i1}^2) \beta_{i3} + 4\beta_{i2}^2 + 6a_{i1} \beta_{i3}) r^2 R_i + 16r^2 \beta_{i4} - 4S_i r \beta_{i4} \\ &+ T_i q_{i\bar{i}}(\beta_{i4} - \beta_{i4})] e^{-4rx} \\ &+ [(4(c_{i1}^1 + c_{i1}^2) \beta_{i2} + 8a_{i1} \beta_{i2} + 4a_{i1} \beta_{i2}) r^2 R_i + 9\beta_{i3} r^2 - 3S_i r \beta_{i3} \\ &+ T_i q_{i\bar{i}}(\beta_{i3} - \beta_{i3})] e^{-3rx} \\ &+ [((c_{i1}^1 + c_{i1}^2) + 2a_{i1}(c_{i1}^1 + c_{i1}^2)) r^2 R_i + 4\beta_{i2} r^2 - 2S_i r \beta_{i2} \\ &+ T_i q_{i\bar{i}}(\beta_{i2} - \beta_{i2})] e^{-2rx} \\ &+ [(c_{i1}^1 + c_{i1}^2)(r^2 - S_i r) + T_i q_{i\bar{i}}[(c_{i1}^1 + c_{i1}^2) - (c_{i1}^1 + c_{i1}^2)]] e^{-rx} \\ &+ T_i q_{i\bar{i}}[(c_{i0}^1 + c_{i0}^2) - (c_{i0}^1 + c_{i0}^2)],\end{aligned}$$

which, in turn, gives

$$\begin{aligned}
 v_{i,3}(x) = & v_{i,2}(x) + \hbar_i 49 R_i \beta_{i7}^2 \frac{e^{-16rx}}{240} + 84 \hbar_i \beta_{i6} \beta_{i7} R_i \frac{e^{-15rx}}{210} \\
 & + \hbar_i (70 \beta_{i5} \beta_{i7} + 36 \beta_{i6}^2) R_i \frac{e^{-14rx}}{182} \\
 & + \hbar_i (56 \beta_{i4} \beta_{i7} + 60 \beta_{i6} \beta_{i5}) R_i \frac{e^{-13rx}}{156} \\
 & + \hbar_i \left(42 \beta_{i3} \beta_{i7} + 48 \beta_{i4} \beta_{i6} + 25 \beta_{i5}^2 \right) R_i \frac{e^{-12rx}}{132} \\
 & + (28 \beta_{i2} \beta_{i7} + 36 \beta_{i3} \beta_{i6} + 40 \beta_{i4} \beta_{i5}) R_i \frac{e^{-11rx}}{110} \\
 & + \hbar_i [(14 \beta_{i7} (c_{i1}^2 + c_{i1}^1) + 24 \beta_{i2} \beta_{i6} + 30 \beta_{i5} \beta_{i3} + 16 \beta_{i4}^2) + 14 a_{i1} \beta_{i7}] R_i \frac{e^{-10rx}}{90} \\
 & + \hbar_i [(12 (c_{i1}^1 + c_{i1}^2) \beta_{i6} + 20 \beta_{i5} \beta_{i2} + 24 \beta_{i3} \beta_{i4} + 12 a_{i1} \beta_{i6}) r^2 R_i + 49 r^2 \beta_{i7} \\
 & - 7 S_i r \beta_{i7} + T_i q_{i\bar{i}} (\beta_{i7} - \beta_{i7})] \frac{e^{-9rx}}{72 r^2} \\
 & + \hbar_i [(10 (c_{i1}^1 + c_{i1}^2) \beta_{i5} + 16 \beta_{i4} \beta_{i2} + 9 \beta_{i3}^2 + 10 a_{i1} \beta_{i5}) r^2 R_i + 36 r^2 \beta_{i6} \\
 & - 6 S_i r \beta_{i6} + T_i q_{i\bar{i}} (\beta_{i6} - \beta_{i6})] \frac{e^{-8rx}}{56 r^2} \\
 & + \hbar_i [(8 (c_{i1}^1 + c_{i1}^2) \beta_{i4} + 12 \beta_{i3} \beta_{i2} + 8 a_{i1} \beta_{i4}) r^2 R_i + 25 r^2 \beta_{i5} - 5 S_i r \beta_{i5} \\
 & + T_i q_{i\bar{i}} (\beta_{i5} - \beta_{i5})] \frac{e^{-7rx}}{42 r^2} \\
 & + \hbar_i [(6 (c_{i1}^1 + c_{i1}^2) \beta_{i3} + 4 \beta_{i2}^2 + 6 a_{i1} \beta_{i3}) r^2 R_i + 16 r^2 \beta_{i4} - 4 S_i r \beta_{i4} \\
 & + T_i q_{i\bar{i}} (\beta_{i4} - \beta_{i4})] \frac{e^{-6rx}}{30 r^2} \\
 & + \hbar_i [(4 (c_{i1}^1 + c_{i1}^2) \beta_{i2} + 8 a_{i1} \beta_{i2} + 4 a_{i1} \beta_{i2}) r^2 R_i + 9 \beta_{i3} r^2 - 3 S_i r \beta_{i3} \\
 & + T_i q_{i\bar{i}} (\beta_{i3} - \beta_{i3})] \frac{e^{-5rx}}{10 r^2} \\
 & + \hbar_i [(c_{i1}^1 + c_{i1}^2) (r^2 - S_i r) + 2 a_{i1} (c_{i1}^1 + c_{i1}^2) r^2 R_i + 4 \beta_{i2} r^2 - 2 S_i r \beta_{i2} \\
 & + T_i q_{i\bar{i}} (\beta_{i2} - \beta_{i2})] \frac{e^{-4rx}}{12 r^2} \\
 & + \hbar_i [(c_{i1}^1 + c_{i1}^2) (r^2 - S_i r) + T_i q_{i\bar{i}} [(c_{i1}^1 + c_{i1}^2) - (c_{i1}^1 + c_{i1}^2)]] \frac{e^{-3rx}}{6 r^2} \\
 & + \hbar_i T_i q_{i\bar{i}} [(c_{i0}^1 + c_{i0}^2) - (c_{i0}^1 + c_{i0}^2)] \frac{e^{-2rx}}{2 r^2} + c_{i0}^3 + c_{i1}^3 e^{-rx}
 \end{aligned}$$

By letting,

$$\begin{aligned}
 \gamma_{i16} &= \frac{\hbar_i 49 R_i \beta_{i7}^2}{240}, \quad \gamma_{i15} = \frac{84 \hbar_i \beta_{i6} \beta_{i7} R_i}{210}, \quad \gamma_{i14} = \frac{\hbar_i (70 \beta_{i5} \beta_{i7} + 36 \beta_{i6}^2) R_i}{182} \\
 \gamma_{i13} &= \frac{\hbar_i (56 \beta_{i4} \beta_{i7} + 60 \beta_{i6} \beta_{i5}) R_i}{156}, \quad \gamma_{i12} = \frac{\hbar_i (42 \beta_{i3} \beta_{i7} + 48 \beta_{i4} \beta_{i6} + 25 \beta_{i5}^2) R_i}{132}
 \end{aligned}$$

$$\begin{aligned}
\gamma_{i11} &= \frac{(28\beta_{i2}\beta_{i7} + 36\beta_{i3}\beta_{i6} + 40\beta_{i4}\beta_{i5}))R_i}{110}, \\
\gamma_{i10} &= \frac{\bar{h}_i[(14\beta_{i7}(c_{i1}^2 + c_{i1}^1) + 24\beta_{i2}\beta_{i6} + 30\beta_{i5}\beta_{i3} + 16\beta_{i4}^2) + 14a_{i1}\beta_{i7}]R_i}{90} \\
\gamma_{i9} &= \frac{\bar{h}_i[(12(c_{i1}^1 + c_{i1}^2)\beta_{i6} + 20\beta_{i5}\beta_{i2} + 24\beta_{i3}\beta_{i4} + 12a_{i1}\beta_{i6})r^2 R_i + 49r^2\beta_{i7} - 7S_i r\beta_{i7} + T_i q_{ii}(\beta_{i7} - \beta_{i7})]}{72r^2} \\
\gamma_{i8} &= \frac{\bar{h}_i[(10(c_{i1}^1 + c_{i1}^2)\beta_{i5} + 16\beta_{i4}\beta_{i2} + 9\beta_{i3}^2 + 10a_{i1}\beta_{i5})r^2 R_i + 36r^2\beta_{i6} - 6S_i r\beta_{i6} + T_i q_{ii}(\beta_{i6} - \beta_{i6})]}{56r^2} \\
\gamma_{i7} &= \frac{\bar{h}_i[(8(c_{i1}^1 + c_{i1}^2)\beta_{i4} + 12\beta_{i3}\beta_{i2} + 8a_{i1}\beta_{i4})r^2 R_i + 25r^2\beta_{i5} - 5S_i r\beta_{i5} + T_i q_{ii}(\beta_{i5} - \beta_{i5})]}{42r^2} + \beta_{i7} \\
\gamma_{i6} &= \frac{\bar{h}_i[(6(c_{i1}^1 + c_{i1}^2)\beta_{i3} + 4\beta_{i2}^2 + 6a_{i1}\beta_{i3})r^2 R_i + 16r^2\beta_{i4} - 4S_i r\beta_{i4} + T_i q_{ii}(\beta_{i4} - \beta_{i4})]}{30r^2} + \beta_{i6} \\
\gamma_{i5} &= \frac{\bar{h}_i[(4(c_{i1}^1 + c_{i1}^2)\beta_{i2} + 8a_{i1}\beta_{i2} + 4a_{i1}\beta_{i2})r^2 R_i + 9\beta_{i3}r^2 - 3S_i r\beta_{i3} + T_i q_{ii}(\beta_{i3} - \beta_{i3})]}{10r^2} + \beta_{i5} \\
\gamma_{i4} &= \frac{\bar{h}_i[(c_{i1}^1 + c_{i1}^2) + 2a_{i1}(c_{i1}^1 + c_{i1}^2)r^2 R_i + 4\beta_{i2}r^2 - 2S_i r\beta_{i2} + T_i q_{ii}(\beta_{i2} - \beta_{i2})]}{12r^2} + \beta_{i4} \\
\gamma_{i3} &= \frac{\bar{h}_i[(c_{i1}^1 + c_{i1}^2)(r^2 - S_i r) + T_i q_{ii}[(c_{i1}^1 + c_{i1}^2) - (c_{i1}^1 + c_{i1}^2)]]}{6r^2} + \beta_{i3}, \\
\gamma_{i2} &= \frac{\bar{h}_i T_i q_{ii}[(c_{i0}^1 + c_{i0}^2) - (c_{i0}^1 + c_{i0}^2)]}{2r^2} + \beta_{i2},
\end{aligned}$$

we obtain

$$v_{i,3}(x) = \sum_{k=2}^{16} \gamma_{ik} e^{-krx} + c_{i0}^1 + c_{i1}^1 e^{-rx} + c_{i0}^2 + c_{i1}^2 e^{-rx} + c_{i0}^3 + c_{i1}^3 e^{-rx},$$

with

$$c_{i1}^3 = \frac{\left(\sum_{k=2}^{16} \gamma_{ik} e^{-krb} - c_{i0}^1 - c_{i1}^1 e^{-rb} - c_{i0}^2 - c_{i1}^2 e^{-rb}\right) - \left(\sum_{k=2}^{16} \gamma_{ik} e^{-kra} - c_{i0}^1 - c_{i1}^1 e^{-ra} - c_{i0}^2 - c_{i1}^2 e^{-ra}\right)}{e^{-ra} - e^{-rb}}$$

$$c_{i0}^3 = - \sum_{k=2}^{16} \gamma_{ik} e^{-kra} - c_{i0}^1 - c_{i1}^1 e^{-ra} - c_{i0}^2 - c_{i1}^2 e^{-ra} - c_{i1}^3 e^{-ra}.$$

An approximate analytical solution truncated to second order would follow as

$$\bar{V}_i(x) \approx v_{i,0}(x) + v_{i,1}(x) + v_{i,2}(x) + v_{i,3}(x) \dots \quad i = 1, 2.$$

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